Quantum critical spin liquids and superconductivity in the cuprates

André LeClair

Newman Laboratory, Cornell University, Ithaca, NY
(Dated: October 2006)

We describe a new kind of quantum critical point in the context of quantum anti-ferromagnetism in 2d that can be understood as a quantum critical spin liquid. Based on the comparison of exponents with previous numerical work, we argue it describes a transition from an anti-ferromagnetic Néel ordered state to a VBS-like state. We argue further that the symplectic fermions capture the proper degrees of freedom in the zero temperature phase that is the parent to the superconducting phase in the cuprates. We then show that our model reproduces some features found recently in experiments and also in the Hubbard model.

INTRODUCTION

Over the last few years there has been much interest in finding new quantum critical points in the context of quantum anti-ferromagnetism in 2d. One motivation are their possible applications to the anti-ferromagnetic phase of the Hubbard model and to superconductivity in the cuprates. Strong arguments were given by Senthil et. al. that there should exist a quantum critical point that separates a Néel order phase from a valence-bond solid like phase[1]. In the model these authors considered, it appears difficult to exhibit this critical point perturbatively. In this paper we consider a different, simpler model which contains a critical point that can easily be studied with a perturbative renormalization group analysis. Our critical exponents agree very favorably with the numerical simulation of the model in [1] carried out by Motrunich and Vishwanath[2], and we interpret this as strong evidence that our model is in the same universality class. In the second part of the paper we take some initial steps toward applying this theory to superconductivity.

For the remainder of this Introduction, we summarize the motivations given in [3] for our model. It is well-known that the continuum limit of the Heisenberg anti-ferromagnet constructed over a Néel ordered state leads to a non-linear sigma model for a 3-component field $\vec{n}(\vec{x})$ satisfying $\vec{n}^2 = 1$. (For a detailed account of this in 1d and 2d, with additional references to the original works see[4].) In 1d a topological term S_θ arises directly in the map to the continuum and affects the low-energy (infra-red (IR)) limit: half-integer spin chains are gapless whereas the integer ones are gapped. This is the well-known Haldane conjecture[5]. It is known from the exact Bethe-ansatz solution of the spin 1/2 chain[6] that the low-lying excitations are spin 1/2 particles referred to a spinons[7].

In 2d one still obtains a non-linear sigma model. The topological term S_θ also arises but, unlike in 1d, it is a renormalization group (RG) irrelevant operator so we discard it. The non-linear constraint $\vec{n}^2 = 1$ renders the model non-renormalizable in 2d. However there is a quantum critical point in the spin system[8, 9] that is

captured by the following euclidean space action:

$$S_{WF} = \int d^3x \left(\frac{1}{2}\partial_\mu \vec{n} \cdot \partial_\mu \vec{n} + \widetilde{\lambda}\, (\vec{n} \cdot \vec{n})^2 \right) \quad (1)$$

$\partial_\mu^2 = \sum_{\mu=1}^3 \partial_{x_\mu}^2$. The fixed point is in the Wilson-Fisher univerality class[10]. The fixed point generalizes to an M component vector $\vec{n}$ and we will refer to the fixed point theory as $O_M^{(D)}$ where $D = d+1$.

Senthil et. al.[1] have given numerous arguments suggesting that anti-ferromagnets can have more exotic quantum critical points that are not in the Wilson-Fisher universality class. They are expected to describe for instance transitions between a Néel ordered state and a valence-bond-solid (VBS)-like phase, and some evidence for such a transition was found by Motrunich and Vishwanath[2]. See also [11]. A large part of the literature devoted to the "deconfined" quantum critical points represent the $\vec{n}$ field as

$$\vec{n} = \chi^\dagger \vec{\sigma} \chi \quad (2)$$

where $\vec{\sigma}$ are the Pauli matrices and $\chi = (\chi_1, \chi_2) = \{\chi_i\}$ is a two component complex *bosonic* spinor. The constraint $\vec{n}^2 = 1$ then follows from the constraint $\chi^\dagger \chi = 1$. Coupling χ to a $U(1)$ gauge field A_μ with the covariant derivative $D_\mu = \partial_\mu - iA_\mu$, then by eliminating the non-dynamical gauge field using it's equations of motion, one can show that the following actions are equivalent:

$$\tfrac{1}{2} \int d^D x \; \partial_\mu \vec{n} \cdot \partial_\mu \vec{n} = \int d^D x \; |D_\mu \chi|^2 \quad (3)$$

Senthil et. al. considered a model where χ was a boson, and added an $F_{\mu\nu}^2$ term which makes the gauge field dynamical.

The central idea of this paper is that the spinon χ is a *fermion*. Numerous arguments were given in[3]. First of all, the equivalence (3) is valid whether χ is a boson or fermion. Secondly, suppose the theory is asymptotically free in the ultra-violet, which it is. Then in this conformally invariant limit, one would hope that the description in terms of $\vec{n}$ or χ have the same numbers of degrees of freedom. One way to count these degrees of freedom is to compute the free energy density at finite

temperature. For a single species of massless particle, the free energy density in $2d$ is

$$\mathcal{F} = -c_3 \frac{\zeta(3)}{2\pi} T^3 \qquad (4)$$

where $c_3 = 1$ for a boson and $3/4$ for a fermion. (ζ is Riemann's zeta function.) (In $1d$ the analog of the above is $\mathcal{F} = -c\pi T^2/6$, where c is the Virasoro central charge[12, 13].) Therefore one sees that the 3 bosonic degrees of freedom of an $\vec{n}$ field has the same c_3 as an $N = 2$ component χ field. One way to possibly understand the change of statistics from bosonic to fermionic is by simply adding a Chern-Simons term to (3)[14].

In analogy with the bosonic non-linear sigma model, since the constraint $\chi^\dagger \chi = 1$ again renders the model non-renormalizable, we relax this constraint and consider the action:

$$S_\chi = \int d^D x \ \left(2\partial_\mu \chi^\dagger \partial_\mu \chi + 16\pi^2 \lambda \ |\chi^\dagger \chi|^2\right) \qquad (5)$$

where now χ is an N-component complex field, sometimes referred to as a symplectic fermion, $\chi^\dagger \chi = \sum_{i=1}^N \chi_i^\dagger \chi_i$. This model may at first appear peculiar, in that the field χ has a Klein-Gordon action but is quantized as a fermion. However, we remind the reader that there is no spin-statistics theorem in $2d$. Note there is no gauge field ("emergent photon") in the model.

As shown in [3], the $(\chi^\dagger \chi)^2$ interactions drive the theory to a new infrared stable fixed point, we refer to as $Sp_N^{(D)}$. For $N = 2$ in $3D$ the exponents were computed to be[3]

$$\eta = 3/4, \quad \nu = 4/5, \quad \beta = 7/10, \quad \delta = 17/7 \quad (Sp_2^{(3)}) \ (6)$$

(The definition of these exponents is given in the next section.) These agree very favorably with the critical exponents found in [2]: $\nu = .8 \pm 0.1$, $\beta/\nu = .85 \pm 0.05$, certainly within error bars. The shift down to $3/4$ from the classical value $\eta = 1$ is entirely due to the fermionic nature of the χ fields. We thus conjecture that the $Sp_2^{(3)}$ model describes a deconfined quantum critical spin liquid.

In the next section we summararize the results of the critical theory found in [3]. In section III we apply this model to high T_c superconductors and describe agreement with some recent experimental results[15].

THE CRITICAL THEORY

The 1-loop beta function for the N component model in D space-time dimensions is

$$\frac{d\lambda}{d\ell} = (4 - D)\lambda + (N - 4)\lambda^2 \qquad (7)$$

where increasing the length e^ℓ corresponds to the flow toward low energies. The above beta function has a zero at

$$\lambda_* = \frac{4 - D}{4 - N} \qquad (8)$$

Note that λ_* changes sign at $N = 4$. It is not necessarily a problem to have a fixed point at negative λ since the particles are fermionic: the energy is not unbounded from below because of the Fermi sea. Near λ_* one has that $d\lambda/d\ell \sim (D - 4)(\lambda - \lambda_*)$ which implies the fixed point is IR stable regardless of the sign of λ_*, so long as $D < 4$. Arguments were given in [3] that at $N = 4$, two of the χ components reconfine into a spin field $\vec{n}$, though we will not need this here.

Definition of the exponents for the $\vec{n}$ field

Though the spinons χ are deconfined, it is still physically meaningful to define exponents in terms of the original order parameter $\vec{n}$, which is represented by eq. (2). These exponents are especially useful if one approaches the fixed point from within an anti-ferromagnetic phase. We then define the exponent η as the one characterizing the spin-spin correlation function:

$$\langle \vec{n}(\mathbf{x}) \cdot \vec{n}(0) \rangle \sim \frac{1}{|\mathbf{x}|^{D-2+\eta}} \qquad (9)$$

For the other exponents we need a measure of the departure from the critical point; these are the parameters that are tuned to the critical point in simulations and experiments:

$$S_\chi \to S_\chi + \int d^D x \ (m^2 \chi^\dagger \chi + \vec{B} \cdot \vec{n}) \qquad (10)$$

Above, m is a mass and $\vec{B}$ the magnetic field. The correlation length exponent ν, and magnetization exponents β, δ are then defined by

$$\xi \sim m^{-\nu}, \qquad \langle \vec{n} \rangle \sim m^\beta \sim B^{1/\delta} \qquad (11)$$

Above $\langle \vec{n} \rangle$ is the one-point function of the field $\vec{n}(\mathbf{x})$ and is independent of $\mathbf{x}$ by the assumed translation invariance.

The above exponents are related to the anomalous dimension of the field χ and the operator $\chi^\dagger \chi$. This leads to the following relations amoung the exponents:

$$\beta = \nu(D - 2 + \eta)/2, \qquad \delta = \frac{D + (2 - \eta)}{D - (2 - \eta)} \qquad (12)$$

The lowest order contributions to the anomalous dimensions of the operator $\chi^\dagger \chi$ arise at 1-loop, and for χ at two loops. The calculation in [3] gives in $3D$:

$$\nu = \frac{2(4 - N)}{7 - N}, \qquad \beta = \frac{2N^2 - 17N + 33}{N^2 - 11N + 28} \qquad (13)$$

For $N = 2$ one obtains the results quoted in the introduction.

It was conjectured in[3] that the $Sp_{-N}^{(3)}$ model has the same fixed point as the $O_N^{(3)}$ model, so that $Sp_{-1}^{(3)}$ is the $3D$ Ising model. The exponents are in very good agreement with known Ising exponents. A shorter version of these results was described in[16].

SUPERCONDUCTIVITY BASED ON SYMPLECTIC FERMIONS

In this section we explain how our quantum critical spin liquid could be relevant to the understanding of superconductivity in the cuprates, which is believed to be a $2 + 1$ dimensional problem[17]. To do this, one must turn to the language of the Hubbard model. In the anti-ferromagnetic phase of the Hubbard model, the spin field $\vec{n} = c^\dagger \vec{\sigma} c$, where c are the physical electrons. Therefore in applying our model to the Hubbard model, the symplectic fermion χ is a descendant of the electron, so it can carry electric charge. Consider the zero temperature phase diagram of the cuprates as a function of the density of holes. At low density there is an anti-ferromagnetic phase. Suppose that the first quantum critical point is a transition from a Néel ordered to a VBS-like phase and is well described by our symplectic fermion model at $N = 2$. Compelling evidence for a VBS like phase has recently been seen by Davis' group[15]; and it in fact resembles more a "VBS spin glass". The superconducting phase actually originates from this VBS-like phase. It is then possible that the 2-component χ fields capture the correct degrees of freedom for the description of this VBS-like phase. These fermionic spin $1/2$ spinon quasi-particles acquire a gap away from the critical point, which is described by the mass term in eq. (10). Note that away from the quantum critical point, the particles already have a gap m because of the relativistic nature of the symplectic fermion[20].

Superconductivity based on the symplectic fermion has some very interesting features. In the VBS-like phase the χ-particles are charged fermions and it's possible that additional phonon interactions, or even the χ^4 interactions that led to the critical theory, could lead to a pairing interaction that causes them to condense into Cooper pairs just as in the usual BCS theory. Recent numerical work on the Hubbard model suggests that the Hubbard interactions themselves can provide a pairing mechanism[18].

Passing to Minkowski space, the hamiltonian of the symplectic fermion is

$$H = \int d^2\mathbf{x} \Big(2\, \partial_t \chi^\dagger \partial_t \chi + 2\vec{\nabla}\chi^\dagger \cdot \vec{\nabla}\chi + m^2 \chi^\dagger \chi + \widetilde{\lambda}\, (\chi^\dagger \chi)^2 \Big) \quad (14)$$

Expand the field in terms of creation/annihilation operators as follows

$$\chi(\mathbf{x}) = \int \frac{d^2\mathbf{k}}{4\pi\sqrt{\omega_\mathbf{k}}} \left(a_\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}} + b_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \right) \quad (15)$$

$$\chi^\dagger(\mathbf{x}) = \int \frac{d^2\mathbf{k}}{4\pi\sqrt{\omega_\mathbf{k}}} \left(a_\mathbf{k}^\dagger\, e^{i\mathbf{k}\cdot\mathbf{x}} + b_\mathbf{k}^\dagger\, e^{-i\mathbf{k}\cdot\mathbf{x}} \right)$$

where $\omega_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}$. Canonical quantization of the χ-fields, $\{\chi^\dagger(\mathbf{x}), \partial_t \chi(\mathbf{x}')\} = i\delta(\mathbf{x} - \mathbf{x}')/2$, leads to the anti-commutation relations

$$\{b_\mathbf{k}^\dagger, b_{\mathbf{k}'}\} = -\{a_\mathbf{k}^\dagger, a_{\mathbf{k}'}\} = \delta_{\mathbf{k},\mathbf{k}'} \quad (16)$$

The free hamiltonian is then

$$H_0 = \int d^2\mathbf{k}\, \omega_\mathbf{k} \left(a_\mathbf{k}^\dagger a_\mathbf{k} + b_\mathbf{k}^\dagger b_\mathbf{k} \right) \quad (17)$$

The minus sign in the anti-commutator of the a's means there are negative norm states in the free Hilbert space[21]. However a simple projection onto even numbers of a-particles gives a unitary Hilbert space. In a potential physical realization, since the anti-ferromagnetic spin field $\vec{n}$ is deconfined, it is clear that the particles come in pairs.

The minus sign in eq. (16) actually leads to a two-band theory. This has been seen experimentally[15] and also in the Hubbard model[18]. There are two kinds of spin $1/2$ particles created by a or b: $a_\mathbf{k}^\dagger|0\rangle = |\mathbf{k}\rangle_a$, $b_\mathbf{k}^\dagger|0\rangle = |\mathbf{k}\rangle_b$, with energies $\varepsilon_{a,b}$:

$$H_0\, |\mathbf{k}\rangle_{a,b} = \varepsilon_{a,b}(\mathbf{k})\, |\mathbf{k}\rangle_{a,b} \quad (18)$$
$$\varepsilon_b(\mathbf{k}) = \omega_\mathbf{k}, \qquad \varepsilon_a(\mathbf{k}) = -\omega_\mathbf{k}$$

Note that $\varepsilon_a \leq -m$ and $\varepsilon_b \geq m$ so there is a gap $2m$. The density of states can easily be computed in the free theory. The density of states per volume is defined so that

$$n = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\, \rho(\mathbf{k}) = \int d\varepsilon\, \rho(\varepsilon) \quad (19)$$

where n is the particle number density. Using $\frac{1}{(2\pi)^2}\int d^2\mathbf{k} = \int d\varepsilon\, \varepsilon/2\pi$, one finds

$$\begin{aligned} \rho(\varepsilon) &= \frac{\varepsilon}{2\pi}\, f_b(\varepsilon) &&\text{for } \varepsilon \geq m \quad (20)\\ &= 0 &&\text{for } -m < \varepsilon < m\\ &= \frac{\varepsilon}{2\pi}\, f_a(\varepsilon) &&\text{for } \varepsilon \leq -m \end{aligned}$$

where $f_{a,b}$ are temperature dependent Fermi-Dirac filling fractions. Interactions will tend to fill the gap.

The last ingredient one needs is a pairing phase transition, so let us turn to the interactions. The $(\chi^\dagger\chi)^2$ interaction is very short ranged since it corresponds to a δ-function potential in position space. Because of the relativistic nature of the fields, the interaction gives rise to a variety of pairing interactions. There are actually

pairing interactions between the two bands. However let us focus on the pairing interactions within each band that resemble BCS pairing. If all momenta have roughly the same magnitude $|\mathbf{k}|$, then the interaction gives the terms (up to factors of π):

$$H_{\text{int}} = -\widetilde{\lambda} \sum_{\mathbf{k},\ i,j=\uparrow,\downarrow} \left(a^{\dagger}_{\mathbf{k},i} a^{\dagger}_{-\mathbf{k},j} a_{-\mathbf{k},i} a_{\mathbf{k},j} + (a \to b) \right) + \dots \quad (21)$$

The overall minus sign of the interaction is due to a fermionic exhange statistics. Because of the overall minus sign this is an attractive pairing interaction as in BCS. One difference is that in addition to the opposite spin pairing interactions with $i \neq j$, there are also equal-spin pairings. The quantum ground state can be further studied by reasonably straightforward application of the mean-field BCS construction[19].

CONCLUSIONS

We have shown that the 2-component relativistic symplectic fermion appears to have some of the right ingredients to explain the zero temperature phase diagram of the high T_c cuprates. It has a quantum critical point that we have interpreted as a transition between an anti-ferromagnetic phase and VBS-like phase. Away from the critical point the quantum spin liquid has a 2-band structure as in the VBS spin-glass phase[15]. It also naturally has BCS-like pairing interactions. A real test of our model would be a measurement of the critical properties of the anti-ferromagnetic to VBS spin-glass phase. The magnetic exponent δ is probably the easiest to measure and our theory predicts $\delta = 17/7$.

ACKNOWLEDGMENTS

[1] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath and M. P. A. Fisher, Phys. Rev. **B70** (2004) 144407, cond-mat/0312617
[2] O. I. Motrunich and A. Vishwanath, Phys. Rev. **B70** (2004) 075104, cond-mat/0311222.
[3] A. LeClair, *Quantum critical spin liquids, the 3D Ising model, and conformal field theory in $2+1$ dimensions,* cond-mat/0610639.
[4] E. Fradkin, *Field Theories of Condensed Matter Systems*, Frontiers in physics vol. 82, Addison-Wesley 1991.
[5] F. D. M. Haldane, Phys. Lett. **93A** (1983) 464; Phys. Rev. Lett. **50** (1983) 1153.
[6] H. Bethe, Zeitschrift für Physik, **71** (1931) 205.
[7] L. D. Faddeev and L. H. Takhtajan, Phys. Lett. **85A** (1981) 375.
[8] S. Chakravarty, B. I. Halperin and D. R. Nelson, Phys. Rev. **B39** (1989) 2344.
[9] A. V. Chubukov, S. Sachdev and J. Ye, Phys. Rev. **B49** (1994) 11919 cond-mat/9304046
[10] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28** (1972) 240; K. G. Wilson and J. Kogut, Phys. Rep. **12** (1974) 75.
[11] A. W. Sandvik, S. Daul, R. R. P. Singh, D. J. Scalapino, Phys. Rev. Lett. **89** (2002) 247201, cond-mat/0205270; A. W. Sandvik and R. Moessner, cond-mat/0507277; A. W. Sandvik, cond-mat/0603179.
[12] J. Cardy, J. Phys. **A17** (1984) 385.
[13] I. Affleck, Phys. Rev. Lett. **56** (1986) 746.
[14] F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (1983) 2250.
[15] Private discussions with S. Davis, results to be published in: Y. Kohsaka et. al., submitted (2006)
[16] A. LeClair, *The 3D Ising and other models from symplectic fermions*, cond-mat/0610817.
[17] P. W. Anderson, Science **235** (1987) 1196.
[18] T. A. Maier, M. S. Jarrell and D. J. Scalapino, cond-mat/0606003, 0608507,
[19] J. R. Schrieffer, *Theory of Superconductivity*, Addison-Wesley, 1964.
[20] Here the "speed of light" is some material dependent quantity such as a Fermi velocity. We set it to 1.
[21] I thank S. Sachdev for pointing out this potential difficulty.